What's the analog of the thm that $\kappa, \tau$ determine a space curve?
Do $H, K$ determine a surface? Fn count? $3-2=1$ (map, reparam). Curves $3-1=2$

rel'n betwn $H$ & $K$?

Difference — canonical coords

Instead, for the simplest analog, ask what determines a parametrized surface / isom.

curves $\longrightarrow \kappa, \tau$, speed unit

Idea – get $\mathrm{Isom}(\mathbb{R}^3)$-inv'ts using a frame (cf. M-C)

frame $F = [\varphi_1 \; \varphi_2 \; N]$

Q: can you always choose $\varphi$ so $F$ is orthonormal?

A: probably not — not locally isometric (why?)

$F^{-1} dF$ matrix of 1-forms do we recognize any?

Recall: $\Gamma_i{}^k{}_j = \Gamma_j{}^k{}_i$

Note $B^i_j = h^i_j$

$\varphi_{ij} = \Gamma_i{}^k{}_j \varphi_k + h_{ij} N$,

this is the defn

$$dF = F \begin{bmatrix} \Gamma_1{}^1{}_i & \Gamma_2{}^1{}_i & -B^1_i \\ \Gamma_1{}^2{}_i & \Gamma_2{}^2{}_i & -B^2_i \\ h_{1i} & h_{2i} & 0 \end{bmatrix} dx^i$$

Can you solve this for $F$? Only if $X_1, X_2$ commute on $\mathbb{R}^2 \times G$

$F^{-1}dF = A \Rightarrow \boxed{dA + A \wedge A = 0}$ ← write out

Converse: $F^i_{\ j}$ as a function on $G$

vector field $X_1$ defined by $D_1 F^i_{\ j} = F^i_{\ k} A^k_{1j}$

$\cancel{A}$ $D_1 = F^i_{\ k} A^k_{1j} \frac{\partial}{\partial F^i_{\ j}} + \frac{\partial}{\partial x^1}$ $A$ just depends on $X$.

$[D_1, D_2] F^i_{\ j} = D_1 (F^i_{\ k} A^k_{2j})$

$= F^i_{\ l} A^l_{2k} A^k_{1j} + F^i_{\ k}$

## Miracle 2

$\Gamma$ determined by Riem metric & its first deriv's

- $\frac{1}{2}\langle \psi_1, \psi_1 \rangle_1 = \langle \psi_{11}, \psi_1 \rangle$ ⟶ recover $\psi_{11}^{\tan}$
- $\langle \psi_1, \psi_2 \rangle_1 = \langle \psi_1, \psi_{21} \rangle + \langle \psi_{11}, \psi_2 \rangle$

  $\quad \frac{1}{2}\langle \psi_1, \psi_1 \rangle_2$

- $\frac{1}{2}\langle \psi_1, \psi_1 \rangle_2 = \langle \psi_{12}, \psi_1 \rangle$ ⟶ recover $\psi_{12}^{\tan}$
- $\frac{1}{2}\langle \psi_2, \psi_2 \rangle_1 = \langle \psi_{12}, \psi_2 \rangle$

**Thm** (Bonnet) Suppose $(E, F, G, e, f, g)$ given on $V \subseteq \mathbb{R}^2$, with $\left\{ \begin{matrix} EG - F^2 > 0 \\ G > 0 \end{matrix} \right\}$, satisfying Gauss + Codazzi. Then $\forall\, q \in V$, $\exists$ nbhd $\mathcal{U}$ of $q$ and $X : \mathcal{U} \to \mathbb{R}^3$ s.t. $X(\mathcal{U})$ has coefficients of I + II given by $(EFG, efg)$. Any 2 differ by rigid motion.

<u>Dfn</u> A connection in $E$ is a map

$$\nabla : \mathrm{Vect}(M) \times C^\infty(E) \longrightarrow C^\infty(E)$$

that is

(i) linear over $C^\infty(M)$ in $X$ $\boxed{\text{spell out}}$

(ii) a derivation over $C^\infty(M)$ in $e$

i.e. $\mathbb{R}$-linear in both and

(i) $\nabla_{fX} e = f \nabla_X e$

(ii) $\nabla_X fe = (Xf)e + f \nabla_X e$

<u>Rmk</u> As we've seen w/ vf's, $d$, etc.

$(\nabla_X e)_p$ depends only on $X_p$, $e$ in a nbhd of $p$.

Difference w/ Lie bracket - tensorial in $X$!

Now we'll build up to seeing why $\Gamma$ describes a connection.

Ex On $E = T\mathbb{R}^n$,

$$\bar\nabla_X Y = X(Y^i)\frac{\partial}{\partial y^i}$$

is the Euclidean connection (check)

Ex Pull-back connection:

$$\begin{tikzcd} \varphi^*E \arrow[d] & E \arrow[d] \\ M \arrow[r, "\varphi"'] & N \end{tikzcd}$$

Recall sections of $\varphi^*E$ are locally $\sum f^a(e_a\circ\varphi)$

$$(\varphi^*\nabla)_X f^a(e_a\circ\varphi) = X(f^a)(e_a\circ\varphi) + f^a(\nabla_{d\varphi X}e_a)\circ\varphi$$

eg $\alpha : I \longrightarrow \mathbb{R}^3$ curve.

$\alpha' = \frac{\partial\alpha}{\partial t} = \sum f^i(t)\frac{\partial}{\partial x^i}$ section of $\alpha^* T\mathbb{R}^3$

$$(\alpha^*\nabla)_{\frac{\partial}{\partial t}}\alpha' = \left(\frac{\partial f^i}{\partial t}\right)\frac{\partial}{\partial x^i} + 0 = \alpha''.$$

Main ex: If $\iota : S \to \mathbb{R}^N$ is ~~a~~ isometric Riemannian submfld, and

$$\pi : \iota^*\mathbb{R}^N \to TS$$

is the orthogonal projection

then $\nabla_X Y := \pi(\iota^*\bar\nabla_X Y)$

is a connection on $TS$

More generally, if $F \subseteq (E,\nabla)$ is any sub-bundle, $\pi$ any proj,

$$\nabla_X f = \pi(\nabla^E_X f).$$

Prove

Connection coefficients (Christoffel symbols)

If $E_\alpha$ basis for $E$, $\frac{\partial}{\partial x^i}$ a coord basis for $TM$, then

$$\nabla_i E_\alpha := \nabla_{\frac{\partial}{\partial x^i}} E_\alpha = E_\beta \Gamma^\beta_{i\ \alpha}$$

eg take $E = TM$, $E_\alpha = E_i$

Warning - not a tensor eg,

$$\nabla_i (E_\alpha g^\alpha_{\ \beta}) = (\nabla_i E_\alpha) g^\alpha_{\ \beta} + E_\alpha g^\alpha_{\ \delta} (g^{-1})^\delta_{\ \gamma} \nabla_i g^\gamma_{\ \beta}$$

$$\nabla_i \tilde{E}_\alpha = E_\gamma \Gamma^\gamma_{i\ \alpha} g^\alpha_{\ \beta} + \tilde{E}_\alpha$$

$$\tilde{\Gamma} = g^{-1} \Gamma g + g^{-1} \nabla g$$

Thurs Conn on tensor bundles,

total covariant derivative

Difference of connections

Next time Miracle 2 in generality.